An Analysis of Adaptive Retransmission Arrays in a Fading Environment

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We analyze in this paper the performance of adaptive retransmission for improving two-way communication between antenna arrays in a randomly fading environment.

For a stationary environment, S. P. Morgan has shown that complex conjugate retransmission reaches a stable state and maximizes the signal-to-noise ratio of a maximal ratio diversity reception system. We show that a simpler system using phase conjugate retransmission will also stabilize and maximize the signal-to-noise ratio of an equal gain diversity reception system.

Where the fading is slow in comparison to the system settling-down time, both systems provide a significant improvement in transmission.

Subject to Rayleigh fading, we have obtained the average signal strength and its cumulative probability distribution for various combinations of numbers of antennas in the two arrays for each of the above mentioned systems. This information is useful in choosing an optimal division of diversity branches for the two antenna arrays. It is further observed that although the phase conjugate retransmission system is much simpler to implement, its performance is only slightly inferior to the corresponding complex conjugate system.

I. INTRODUCTION

Adaptive antenna arrays have been the subject of numerous investigations.[1–3] In an adaptive transmitting array, the individual element is excited according to information derived from the incident pilot field. For example, in a *complex conjugate* system, the excitation currents are proportional to the complex conjugate of the incident voltages while the total power radiated is kept constant. In a *phase conjugate* system, the currents are kept constant while the phases are adjusted according to the conjugate phase of the incident voltages.

In a free-space environment, that is, plane wave incident from a particular direction, it is well known tbat phase reversal would steer the radiated heam toward the source antenna. Cutler and others[2] have shown how phase reversal can be acbieved hy frequency conversion of the pilot signal.

The role of adaptive retransmission in a multipatb fading environment, for example, mobile radio, tropscatter communication, and so on, has received far less attention. Still unanswered is the question of whether the phase conjugate or the complex conjugate retransmission schemes could improve the communication link and reach a stable state. In his work, S. P. Morgan has shown that, in a stationary arbitrary environment, stable state and maximal power transfer can be achieved hy complex conjugate retransmission.[3]

In this paper, we show that the much simpler phase conjugate system will also reacb a stable state. Furthermore, assuming equal amplitude transmitting currents on the antenna elements, the summation of voltages received at one array is equal to that of the other array and is maximized. Consequently, the phase conjugate retransmission system will maximize the signal-to-noise ratio (S/N) of an equal gain diversity reception system.[4]

In general, the fundamental differences of the two retransmission schemes are that the phase conjugate retransmission maximizes the sum of the amplitudes of the voltages received and tbe complex conjugate retransmission maximizes the total power received.

Where fading is slow in comparison to the time required to reacb an equilibrium state, both systems could be used to improve the quality of a fading communication link.

We investigate the performance of these two systems in actual fading environments. In particular, we want to know bow these two systems differ in average S/N, what the S/N prohability distributions are, how much they improve fading statistics over a single branch system and, finally, what the optimal division of numher of antennas would be between the two antenna arrays.

In order to answer these questions, we must first establish the characteristics of the medium which links the two antenna arrays. For example, in a mobile radio the signal received by a single antenna is rapid varying and can he characterized by Rayleigh statistics over distances of a few hundred wavelengths.[5] However, over an extended range of observations, other large-scale phenomena such as distance variations, shadowing, and channeling hy streets will produce slow variations of the average signal strength received. The adaptive

retransmission system per se can reduce the rapid fluctuations but will be of little help in reducing those long-term variations. Consequently, the comparison of the performance of adaptive retransmission arrays will be based on their relative effectiveness in reducing the rapid Rayleigh fading.

The Rayleigh fading is also an excellent approximation in other communication systems such as long-range UHF and SHF tropospheric transmission,[4] and so on. Furthermore, results obtained from Rayleigh fading can give significant insight into the performance of adaptive antenna arrays under other fading conditions.

Based on Rayleigh fading statistics, we investigated the cumulative probability distribution (CPD) of the signal strength of an m:n array system. By m:n we mean that there are m antennas at station 1 and n antennas at station 2. The analysis is done by the Monte Carlo method on a digital computer. The 99 percent reliability level* as well as the average signal strength for a unity transmitter power are obtained. It is interesting to note that with the help of interpolation, in most cases, only 96 computer samples are sufficient to yield a CPD which is accurate up to a few tenths of a dB for all the information we need.

The average S/N of the two retransmission schemes are compared. It is observed that although the phase conjugate system is much simpler to build, it is only slightly inferior to the complex conjugate retransmission system.

For other types of fading distributions, the techniques described here can readily be applied.

II. ANALYSIS OF THE PHASE CONJUGATE RETRANSMISSION

The configuration of the arrays is depicted in Fig. 1. The open circuit voltages and the transmitting currents in each array are represented by column vectors with the time factor exp $(j\omega t)$ suppressed. The mutual couplings are neglected and the antennas in each array are assumed to be identical, with input resistance R during transmission and admittance G during reception.

The transmitting current vector I_2 at array 2 produces the received voltage vector at array 1,

$$V_1 = C\Gamma I_2 \tag{1}$$

where Γ is an $m \times n$ matrix whose elements are proportional to the

* The 99 percent reliability level is defined such that for 99 percent of the time the signal strength is above this level.

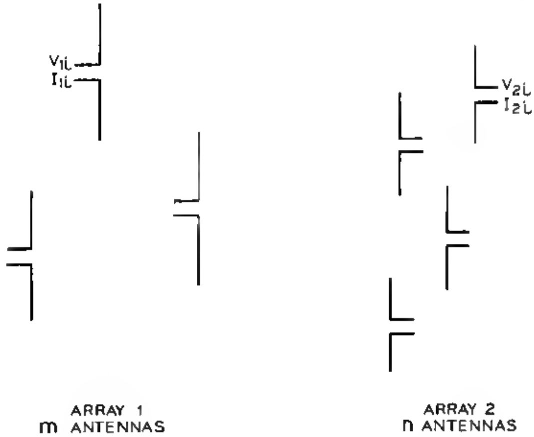

Fig. 1—Arrays in adaptive retransmission system.

transmission between a particular pair of antennas. The real constant C stands for the average transmission loss.

By reciprocity, the received voltage at array 2 is,

$$V_2 = C\Gamma^t I_1 \tag{2}$$

where the superscript t stands for the transpose of the Γ matrix.

Here according to our definition of phase conjugate retransmission, the elements of I_1 and I_2 are of unity amplitudes although their phases could be different. Multiplying equations (1) and (2) by I_1 and I_2, respectively, we obtain the following

$$\langle V_1 , I_1 \rangle = C\langle \Gamma I_2 , I_1 \rangle, \tag{3}$$

$$\langle V_2 , I_2 \rangle = C\langle \Gamma^t I_1 , I_2 \rangle \tag{4}$$

where the brackets $\langle\ \rangle$ stand for inner product. Equations (3) and (4) are equal, and we obtain the following reciprocity relation

$$\langle V_1 , I_1 \rangle = \langle V_2 , I_2 \rangle. \tag{5}$$

2.1 *Stabilization of the Phase Conjugate Retransmission System*

Let array 1 be excited initially with current I_1 which produces V_2 at array 2. And let array 2 be excited with I_2 which produces V_1 at array 1. Equation (5) holds and we have the following

$$\sum_{i=1}^{m} V_{1i} I_{1i} = \sum_{i=1}^{n} V_{2i} I_{2i} \tag{6}$$

where the subscript i stands for the ith element of the array.

Consider now the excitation at array 2. Since the I_{2i}'s are of unity amplitude, the quantity $\sum_{i=1}^{n} V_{2i}I_{2i}$ can be maximized by choosing I'_{2i} to be phase conjugate to V_{2i}. We shall call this real maximum quantity λ. Let V'_1 be the voltage vector produced by I'_{2i}; then we have

$$\sum_{i=1}^{m} V'_{1i}I_{1i} = \sum_{i=1}^{n} V_{2i}I'_{2i} = \sum_{i=1}^{n} |V_{2i}| = \lambda. \tag{7}$$

Let us now consider the excitation of array 1. Obviously the quantity $\sum_{i=1}^{m} V'_{1i}I_{1i}$ can be maximized if we choose I'_{1i} to be the phase conjugate of V'_{1i}. It then follows that

$$\sum_{i=1}^{m} V'_{1i}I'_{1i} = \sum_{i=1}^{m} |V'_{1i}| = \lambda' \geqq \lambda. \tag{8}$$

Let V'_{2i} be the voltages produced by I'_{1i}. We obtain, by applying equation (6), the following,

$$\sum_{i=1}^{m} V'_{1i}I'_{1i} = \sum_{i=1}^{n} V'_{2i}I'_{2i} = \lambda' \geqq \lambda. \tag{9}$$

Now I''_{2i} can again be chosen to be phase conjugate to V'_{2i} and we obtain

$$\sum_{i=1}^{m} V'_{2i}I''_{2i} = \sum_{i=1}^{n} |V'_{2i}| = \lambda'' \geqq \lambda' \geqq \lambda. \tag{10}$$

This process continues with each new choice of I representing the actual retransmission adjustment made by the antenna system. It is obvious from equation (10) that each retransmission yields a new value of λ which is real and bigger than or equal to the previous value. However, because of the finite number of antennas involved, λ cannot increase indefinitely. The iteration process must therefore finally settle down to a value λ_f which no longer changes. If this is so, we have

$$\sum_{i=1}^{m} V^f_{1i}I^f_{1i} = \sum_{i=1}^{n} V^f_{2i}I^f_{2i} = \lambda_f. \tag{11}$$

The fact that λ_f is real, and also that we cannot vary the phase of I^f_{2i} and I^f_{1i} to make λ_f larger automatically guarantees that I^f_{1i} and I^f_{2i} are phase conjugate to V^f_{1i} and V^f_{2i}, respectively. In this case, our phase conjugate retransmission apparatus will no longer change the phases of I^f_{1i} and I^f_{2i} because they have already reached their proper value. Therefore, we have arrived at a stable state. In this case equation (11) can be further simplified to

$$\sum_{i=1}^{m} |V^f_{1i}| = \sum_{i=1}^{n} |V^f_{2i}| = \lambda_f. \tag{12}$$

So far we have demonstrated that each retransmission tends to increase λ and a stable state must finally be reached. It still remains to be shown that this stable state yields the absolute maximum λ. It is quite possible that several pairs of I_1 and I_2 exist such that they are phase conjugate to V_1 and V_2 but their corresponding λ_f's are different. This is similar to the existence of different eigenstates in matrix analysis. As is well known in matrix algebra, unless the initial vector is orthogonal to the maximum eigenstate, we would invariably obtain the maximum eigenstate through iterations.

Since the phase conjugate operation on V to produce I is a nonlinear operation, an analytical analysis along the above lines is extremely difficult, if not impossible. However, in the next section we show with computer simulation that the phase conjugate retransmission process converges rapidly and the probability of ending up in a nonmaximum state of λ_f is practically zero.

2.2 *Computer Simulation*

The convergence test was done by choosing a 3 : 4 array system as a particular trial case. We started by arbitrarily choosing a Γ matrix, which was defined by $\Gamma_{IJ} = I/1.2 + J/2 - 1 + j[I/2.3 + 2 - J/1.2]$. The initial values of I_1 were chosen such that,

$$I_1 = [1, \exp(j\theta), \exp(j\phi)]. \tag{13}$$

The phase angles θ and ϕ were allowed to run through 0 to 2π, in 10 equal steps. Therefore, we had 100 different initial trial values of I_1. For each initial set of I_1, we calculated V_2 produced and formed I_2 which produced V_1. I_1 was then readjusted according to the V_1 just produced. In each retransmission, we also computed the quantity λ. It was observed that in all these one hundred trials, the currents and λ approached their specific final values within a few retransmissions. For this particular choice of Γ, $\lambda_f = 31.3719$. The first value of λ obtained, that is, $\sum_{i=1}^{4} | V_{2i} |$, was always smaller than λ_f but after the first retransmission, it invariably came very close to λ_f. For example, in one case the first λ was 10.72; after retransmission at array 2 we obtained a λ of 30.73 at array 1. After this array retransmitted back to array 2, the value agreed with λ_f to the fourth decimal place.

Next we tried to determine if λ_f is the absolute maximum. In other words, we wanted to check if λ_f was bigger than the λ, that is, $\sum_{i=1}^{4} V_{2i}$, produced by any arbitrary I_1. This survey was done by varying θ and ϕ in 50 steps from 0 to 2π. Computation indicated that all the 2500 values of λ produced were smaller than λ_f and that λ_f was indeed the real maximum.

A similar test was performed on a 4 : 5 array system and we obtained similar results as reported for the 3 : 4 system. In the 4 : 5 array system, the Γ_{IJ} were defined as $(I - J)/3 + I^2J/6 - 5 + j[(I - I^2 + J)/1.4 + 3.5]$.

III. SIGNAL-TO-NOISE RATIO

Let V_{1i} be the voltage response at the ith elementary antenna. Furthermore, let η_{1i} be the corresponding noise voltage which satisfies,

$$\langle \eta_{1i}\eta_{1j}\rangle_{av} = \begin{cases} N^2 & i = j, \\ 0 & i \neq j \end{cases} \tag{14}$$

where the $\langle\ \rangle_{av}$ stand for time average.

3.1 *S/N of Phase Conjugate System Using Equal Gain Diversity Combining Technique*

The S/N of an m-branch diversity equal gain system is,

$$\mathrm{S/N} = \left[\sum_{i=1}^{m} |V_{1i}|\right]^2 \Big/ mN^2 = \lambda_f^2/mN^2. \tag{15}$$

Recall that there are n elements at the other array, which radiates a total power to the amount of nR, therefore the S/N of the received signal per unit power radiated is,

$$\mathrm{S/N} = \lambda_f^2/nmN^2R. \tag{16}$$

It is therefore obvious that the S/Ns at both arrays are identical.

3.2 *S/N of Complex Conjugate System Using Maximal Ratio Diversity Combining Technique*

The excitation currents of a complex conjugate retransmission system are related to the incoming voltages by,

$$I_2 = K_2V_2^*, \tag{17}$$

$$I_1 = K_1V_1^* \tag{18}$$

where K_1 and K_2 are scalars to keep the total radiated power constant. For unity transmitter power, the received power at arrays 1 and 2 are maximized and are equal,[3]

$$P_{1R} = P_{2R} = \frac{G}{R}C^2\lambda_m \tag{19}$$

where λ_m is the maximum eigenvalue of the hermitian matrix $\Gamma\Gamma^+$.

The validity of equation (19) is subject to the constraint that when the adaptive retransmission array starts operation, its current vector should not be orthogonal to the maximum eigenvector of the $\Gamma\Gamma^{+}$ matrix. The S/N of a multibranch maximal ratio reception system then is,

$$\mathrm{S/N} = \frac{C^2}{\mathrm{RN}^2}\lambda_m. \tag{20}$$

It can be seen that the S/Ns at both arrays are equal.

IV. EVALUATION OF THE CUMULATIVE PROBABILITY DISTRIBUTION

The complexity of tbe quantities λ_m and λ_f makes a closed form solution of the CPD extremely difficult, if not impossible. Therefore, we try instead the Monte Carlo method and aim at a numerical solution. The essence of the method is to choose for each element of the Γ matrix a random variable of the form $u + jv$. The variables u and v, according to our assumption of independent Rayleigh fading statistics, are normalized independent gaussian variables. For a particular $m{:}n$ array system, we can therefore evaluate the maximum eigenvalue λ_m by repeated matrix multiplication.[6] The value λ_f is evaluated by iterations according to the retransmission schemes defined in Section 2.2.

The computed values of λ_m *and* λ_f are stored. Then we start the wholo process again by choosing elements for another Γ matrix and evaluate the corresponding λ_m and λ_f. The CPD curves are developed after a sufficient number of calculations.

Two tests of convergence are made. The first is the comparison of the calculated CPD curves of variables $| u_1 + jv_1 |^2$ or $| u_1 + jv_1 |^2 + | u_2 + jv_2 |^2$ to that of the known theoretical curves. It is understood here that u's and v's refer to independent normalized gaussian random variables. Hence, these curves represent respectively the CPD of maximal reception of single or two-channel Rayleigh signals.[4]

The results are presented in Fig. 2. A close look at Fig. 2 indicates that as far as the 99 percent reliability and the average signal levels are concerned, 900 sample points are sufficient for a single Rayleigh and 300 sample points for two Rayleighs.

A second test is made on the 2:2 and 2:4 antenna system and is shown in Fig. 3. The dB scale is chosen such that the average S/N of a single Rayleigh variable, that is, the received S/N of a 1:1 array system, is at 0 dB. It is observed that 96 samples are already sufficient

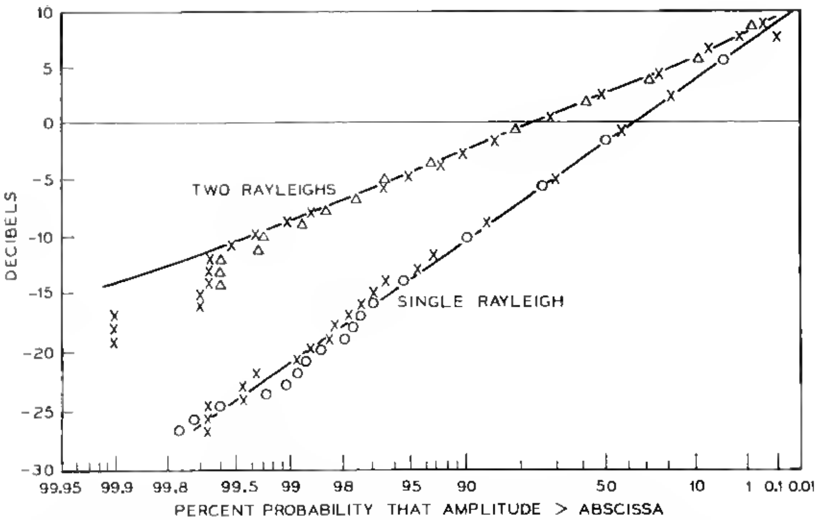

Fig. 2—Comparison of Monte Carlo method and theoretical calculation, △, 300 samples; x, 900 samples; o, 1800 samples; ———, theoretical curve.

to yield what we want since these points lie very close to the curve drawn through the points computed from 900 samples. With the required sample points greatly reduced to this number, it is possible to make a fast and inexpensive check of an extensive combination of m:n arrays.

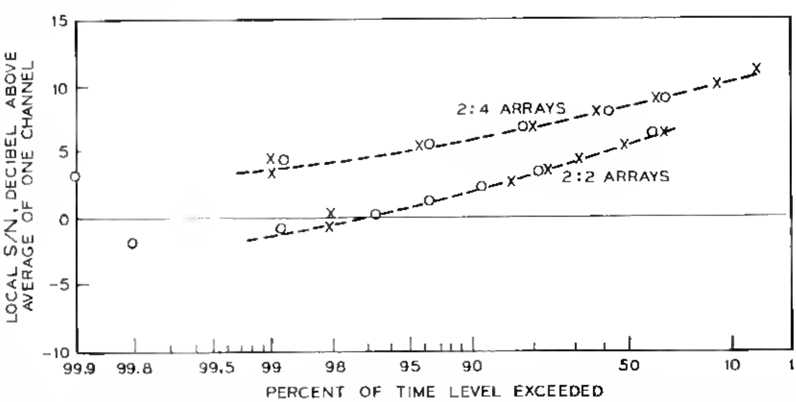

Fig. 3—Comparison of 96 and 960 samples. Complex conjugate retransmission maximal ratio reception. o, 960 points; x, 96 points; — — —, curve fitted to 960 points.

V. DISCUSSION OF NUMERICAL RESULTS

We look at the complex conjugate retransmission system first. Incorporated with maximal ratio diversity reception, this system provides the best S/N performance obtainable from a particular $m{:}n$ array system.

The average S/N is presented in Fig. 4. It is seen that for small numbers of n, there do exist appreciable improvements in average signal level as m changes from 1 to 4. However, as n increases the advantage diminishes. For example, a 1:50 array has the same average signal level as 2:44, 3:39, and 4:35 arrays. This is in sharp contrast to the case of adaptive arrays with nonfading signals. In that case, plane wave incidence is assumed and an $m{:}n$ array would have the same S/N as a $1{:}mn$ array (Fig. 4).

A simple explanation of the difference between the fading and the nonfading arrays is the following: In both cases, the $1{:}mn$ adaptive retransmission system guarantees that the voltages produced by the mn elements at the single array add in phase. In the $m{:}n$ system, the

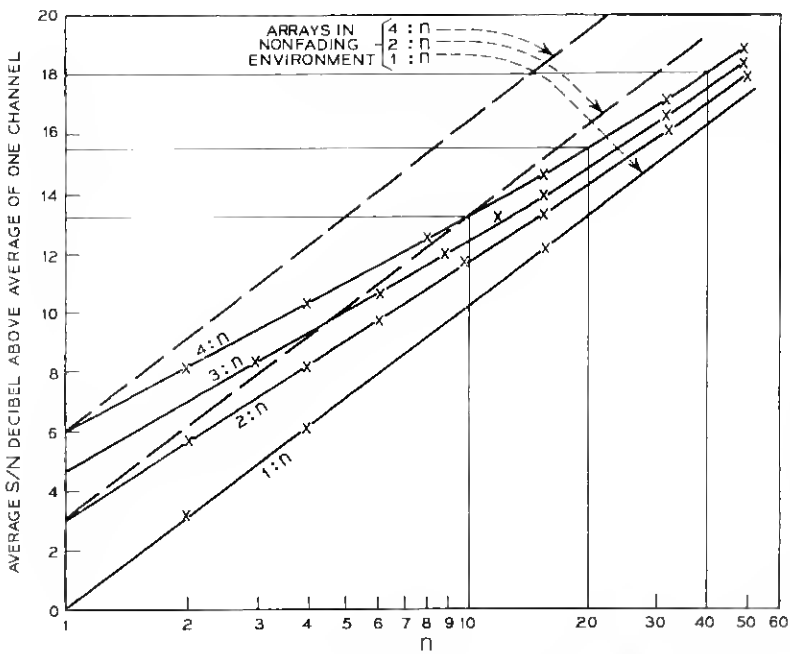

Fig. 4—Average S/N of complex conjugate retransmission arrays.

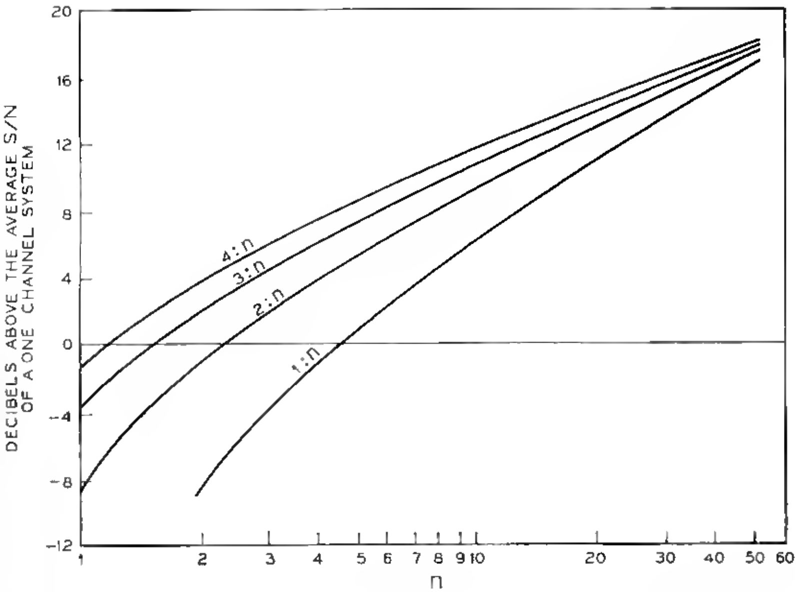

Fig. 5—99 percent reliability level. Complex conjugate retransmission maximal ratio diversity reception.

voltage components produced by the n antennas again add in phase at each antenna of the "m" array if plane wave incidence is assumed. Consequently, the power received is identical to that of the $1{:}mn$ array. However, in a random environment the n voltages components at each antenna element in the m array no longer add in phase; therefore, the $m{:}n$ system receives less power than that of the $1{:}mn$ system.

With reference to Fig. 2, we notice that for 99 percent of the time, the single Rayleigh signal has a value above -20.6 dB; we will designate -20.6 dB as the 99 percent reliability level. Hence the difference in dB values of two antenna systems for a particular reliability indicates their difference in signal threshold or their difference in the required transmitter power. The 99 percent reliability level is presented in Fig. 5. We next define fading range as the dB difference between the average S/N and the 99 percent reliability level. Therefore, fading range should provide a good indication of the smoothness of the received signal. The fading range is presented in Fig. 6. It is seen that as n increases, the 99 percent reliability level approaches the average signal level. In other words this means that as

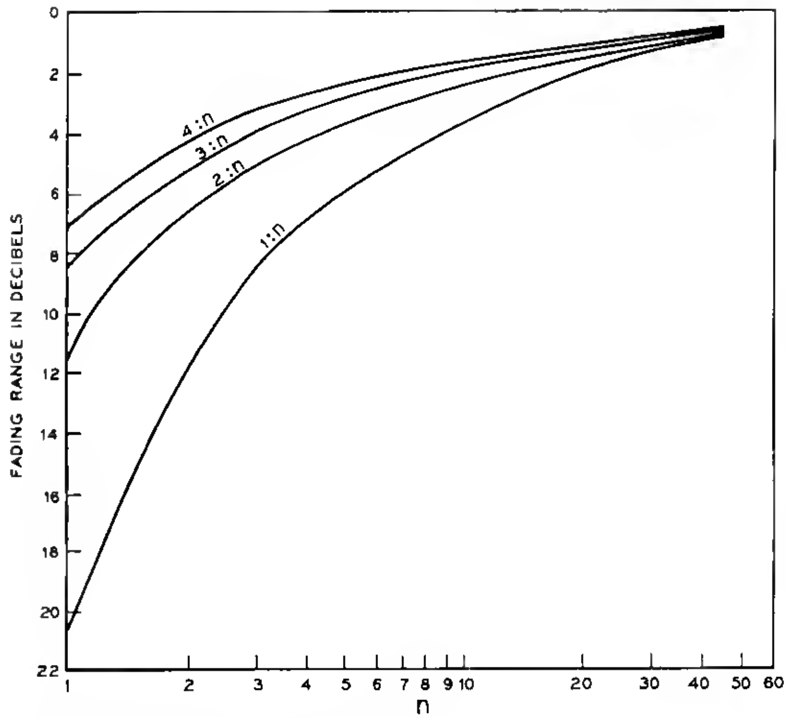

Fig. 6—Fading range of an m:n array system. Complex conjugate retransmission maximal ratio diversity reception.

the number of diversity branches increases, the fading range starts to diminish. Figure 7 presents the CPD of a 4:32 array system. We note that the CPD curve is extremely flat and the signal level varies within a ±1 dB range, indicating a greatly reduced fading range as compared to either Figs. 2 or 3.

We discuss now results obtained from the phase conjugate retransmission system. In this system, as was discussed in Section II, the S/N, of an equal gain diversity reception system is maximized. It is observed that because of this maximization effect, the performance of the phase conjugate system is not much inferior to that of the complex conjugate system. For example, the CPDs of the S/N for both systems in the case of a 2:4 array system are presented in Fig. 8. The CPD curves of the two systems differ approximately by the average S/N difference. Therefore, the difference in average S/N

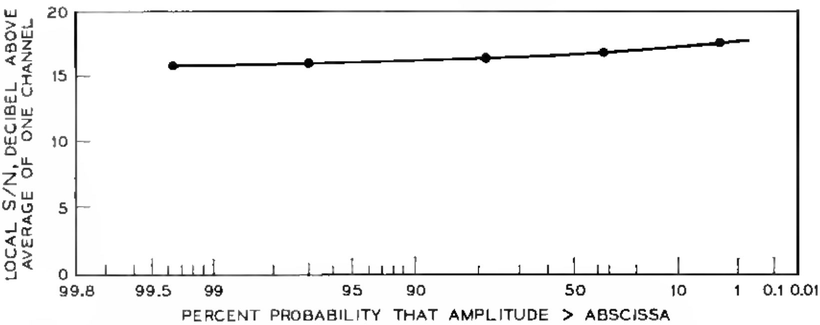

Fig. 7—CPD of a 4:32 array system. Complex conjugate retransmission maximal ratio diversity reception.

of the two systems is also a good indication of their difference in percentile reliability levels.

The average S/N of the two systems is shown in Fig. 9 for $2{:}n$ and $4{:}n$ array systems. It is seen that for the same $m{:}n$ array, the difference of the two systems is small, that is, within a dB or so.

VI. CONCLUSIONS

We observed that in a fading environment, both complex conjugate retransmission and phase conjugate retransmission systems are capable of reaching a stable state and yield optimum results by greatly increasing the S/N at the receiving stations.

The performance of these two systems differs little. Therefore the

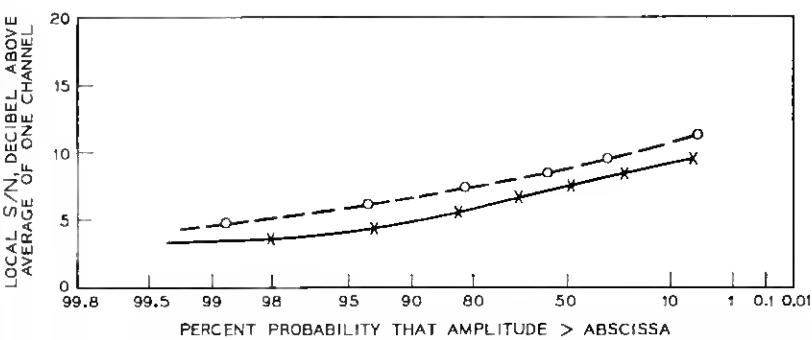

Fig. 8—CPD curves of a 2:4 array system. o, complex conjugate retransmission with maximal ratio reception; x, phase conjugate retransmission with equal gain reception.

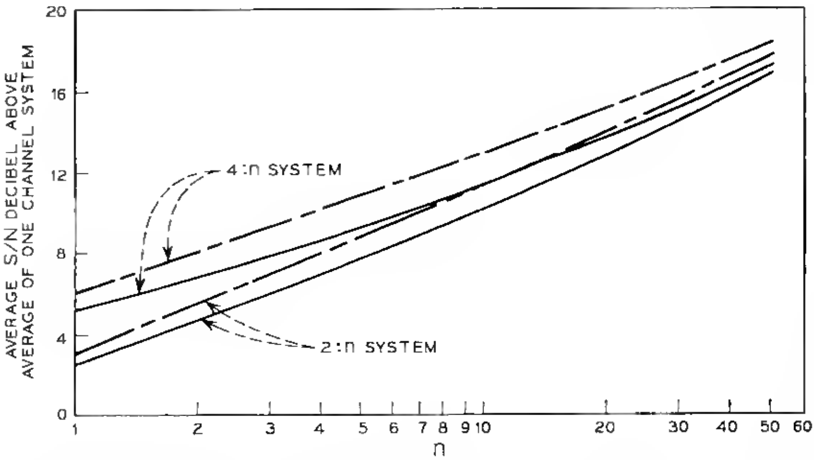

Fig. 9—Average S/N of antenna systems $2:n$ and $4:n$ — — — —, complex conjugate maximal ratio reception; ————, phase conjugate retransmission equal gain reception.

choice of a particular scheme should be based on practical considerations. For example, in the phase conjugate system, the total power is divided equally among all the antenna elements. On the other hand, the complex-conjugate retransmission system requires that the total power be distributed in a complicated fashion. In practice this means that each antenna-feeding appartus must be equipped to handle power far exceeding that of the phase conjugate system.

In view of the simplicity of the phase conjugate retransmission compared to the complex conjugate retransmission (which must keep the total power transmitted constant), and only slightly inferior performance, tbe former appears to be a more attractive system.

As far as the division of diversity branches is concerned, it can be seen from Fig. 4 that for small numbers of antennas, an $m:n$ array would have similar performance to an $mn:1$ array. However, as the number of elements involved becomes larger, this relation no longer holds. For example the performance of a $4:n$ array would approach a $1:n$ array as n increases indefinitely.

VII. ACKNOWLEDGMENT

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